

Sparse Recovery with Very Sparse Compressed Counting

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Abstract

Compressed¹ sensing (sparse signal recovery) often encounters nonnegative data (e.g., images). Recently [11] developed the methodology of using (dense) *Compressed Counting* for recovering nonnegative K -sparse signals. In this paper, we adopt *very sparse Compressed Counting* for nonnegative signal recovery. Our design matrix is sampled from a maximally-skewed α -stable distribution ($0 < \alpha < 1$), and we sparsify the design matrix so that on average $(1 - \gamma)$ -fraction of the entries become zero. The idea is related to *very sparse stable random projections* [9, 6], the prior work for estimating summary statistics of the data.

In our theoretical analysis, we show that, when $\alpha \rightarrow 0$, it suffices to use $M = \frac{K}{1-e^{-\gamma K}} \log N/\delta$ measurements, so that with probability $1 - \delta$, all coordinates can be recovered within ϵ additive precision, in one scan of the coordinates. If $\gamma = 1$ (i.e., dense design), then $M = K \log N/\delta$. If $\gamma = 1/K$ or $2/K$ (i.e., very sparse design), then $M = 1.58K \log N/\delta$ or $M = 1.16K \log N/\delta$. This means the design matrix can be indeed very sparse at only a minor inflation of the sample complexity.

Interestingly, as $\alpha \rightarrow 1$, the required number of measurements is essentially $M = eK \log N/\delta$ provided $\gamma = 1/K$. It turns out that this complexity $eK \log N/\delta$ (at $\gamma = 1/K$) is a general worst-case bound.

¹Part of the content of this paper was submitted to a conference in May 2013.

1 Introduction

In a recent paper [11], we developed a new framework for compressed sensing (sparse signal recovery) [4, 2], by focusing on nonnegative sparse signals, i.e., $\mathbf{x} \in \mathbb{R}^N$ and $x_i \geq 0, \forall i$. Note that real-world signals are often nonnegative. The technique was based on *Compressed Counting (CC)* [8, 7, 10]. In that framework, entries of the (dense) design matrix are sampled i.i.d. from an α -stable maximally-skewed distribution. In this paper, we integrate the idea of *very sparse stable random projections* [9, 6] into the procedure, to develop *very sparse compressed counting for compressed sensing*.

In this paper, our procedure for compressed sensing first collects M non-adaptive linear measurements

$$y_j = \sum_{i=1}^N x_i [s_{ij} r_{ij}], \quad j = 1, 2, \dots, M \quad (1)$$

Here, s_{ij} is the (i, j) -th entry of the design matrix with $s_{ij} \sim S(\alpha, 1, 1)$ i.i.d, where $S(\alpha, 1, 1)$ denotes an α -stable maximally-skewed (i.e., skewness = 1) distribution with unit scale. Instead of using a dense design matrix, we randomly sparsify $(1 - \gamma)$ -fraction of the entries of the design matrix to be zero, i.e.,

$$r_{ij} = \begin{cases} 1 & \text{with prob. } \gamma \\ 0 & \text{with prob. } 1 - \gamma \end{cases} \quad i.i.d. \quad (2)$$

And any s_{ij} and r_{ij} are also independent.

In the decoding phase, our proposed estimator of the i -th coordinate x_i is simply

$$\hat{x}_{i,min,\gamma} = \min_{j \in T_i} \frac{y_j}{s_{ij} r_{ij}} \quad (3)$$

where T_i is the set of nonzero entries in the i -th row of the design matrix, i.e.,

$$T_i = \{j, 1 \leq j \leq M, r_{ij} = 1\} \quad (4)$$

Note that the size of the set $|T_i| \sim \text{Binomial}(M, \gamma)$.

To analyze the sample complexity (i.e., the required number of measurements), we need to study the following error probability

$$\mathbf{Pr}(\hat{x}_{i,min,\gamma} > x_i + \epsilon) \quad (5)$$

from which we can derive the sample complexity by using the following inequality

$$N \mathbf{Pr}(\hat{x}_{i,min,\gamma} > x_i + \epsilon) \leq \delta \quad (6)$$

so that any x_i can be estimated within $(x_i, x_i + \epsilon)$ with a probability (at least) $1 - \delta$.

Main Result 1: As $\alpha \rightarrow 0+$, the required number of measurements is

$$M = \frac{1}{-\log \left[1 - \frac{1}{K+1} (1 - (1 - \gamma)^{K+1}) \right]} \log N / \delta \quad (7)$$

which can essentially be written as

$$M = \frac{K}{1 - e^{-\gamma K}} \log N / \delta \quad (8)$$

If $\gamma = 1/K$, then the required M is about $1.58K \log N/\delta$. If $\gamma = 2/K$, then M is about $1.16K \log N/\delta$. In other words, we can use a very sparse design matrix and the required number of measurements will only be inflated slightly, if we choose to use a small α .

Indeed, using $\alpha \rightarrow 0+$ achieves the smallest complexity. However, there will be a numerical issue if α is too small. To see this, consider the approximate mechanism for generating $S(\alpha, 1, 1)$ by using $1/U^{1/\alpha}$, where $U \sim \text{unif}(0, 1)$. If $\alpha = 0.05$, then we have to compute $(1/U)^{20}$, which may potentially create numerical problems. In our Matlab simulations, we do not notice obvious numerical issues with $\alpha = 0.05$ (or even smaller). However, if a device (e.g., camera or other hand-held device) has a limited precision and/or memory, then we expect that we must use a larger α , away from 0.

Main Result 2: If $x_i > \epsilon$ whenever $x_i > 0$, then as $\alpha \rightarrow 1-$, the required number of measurements is

$$M = \frac{1}{-\log \left(1 - \frac{1}{K+1} \left(1 - \frac{1}{K+1} \right)^K \right)} \log N/\delta, \quad \text{with } \gamma = \frac{1}{K+1} \quad (9)$$

This complexity bound can essentially be written as

$$M = eK \log N/\delta, \quad \text{with } \gamma = \frac{1}{K} \quad (10)$$

Interestingly, this result $eK \log N/\delta$ (with $\gamma = 1/K$) is the general worse-case bound.

2 A Simulation Study

We² consider two types of signals. To generate “binary signal”, we randomly select K (out of N) coordinates to be 1. For “non-binary signal”, we assign the values of K randomly selected nonzero coordinates according to $|N(0, 5^2)|$. The number of measurements is determined by

$$M = \nu K \log N/\delta \quad (11)$$

where $N \in \{10000, 100000\}$, $\delta = 0.01$ and $\nu \in \{1.2, 1.6, 2\}$. We report the normalized recovery errors:

$$\text{Normalized Error} = \sqrt{\frac{\sum_{i=1}^N (x_i - \text{estimated } x_i)^2}{\sum_{i=1}^N x_i^2}} \quad (12)$$

We experiment with all possible values of $1/\gamma \in \{1, 2, 3, \dots, K\}$, although we only plot a few selected γ values in Figures 1 to 4. For each combination (γ, N, ν) , we conduct 100 simulations and report the median errors. The results confirm our theoretical analysis. When ν is small (i.e., less measurements), we need to choose a small α in order to achieve perfect recovery. When ν is large (i.e., more measurements), we can use a larger α . Also, the simulations confirm that, in general, we can choose a very sparse design.

²This report does not include comparisons with the SMP algorithm [1, 5], as we can not run the code from http://groups.csail.mit.edu/toc/sparse/wiki/index.php?title=Sparse_Recovery_Experiments, at the moment. We will provide the comparisons after we are able to execute the code. We thank the communications with the author of [1, 5].

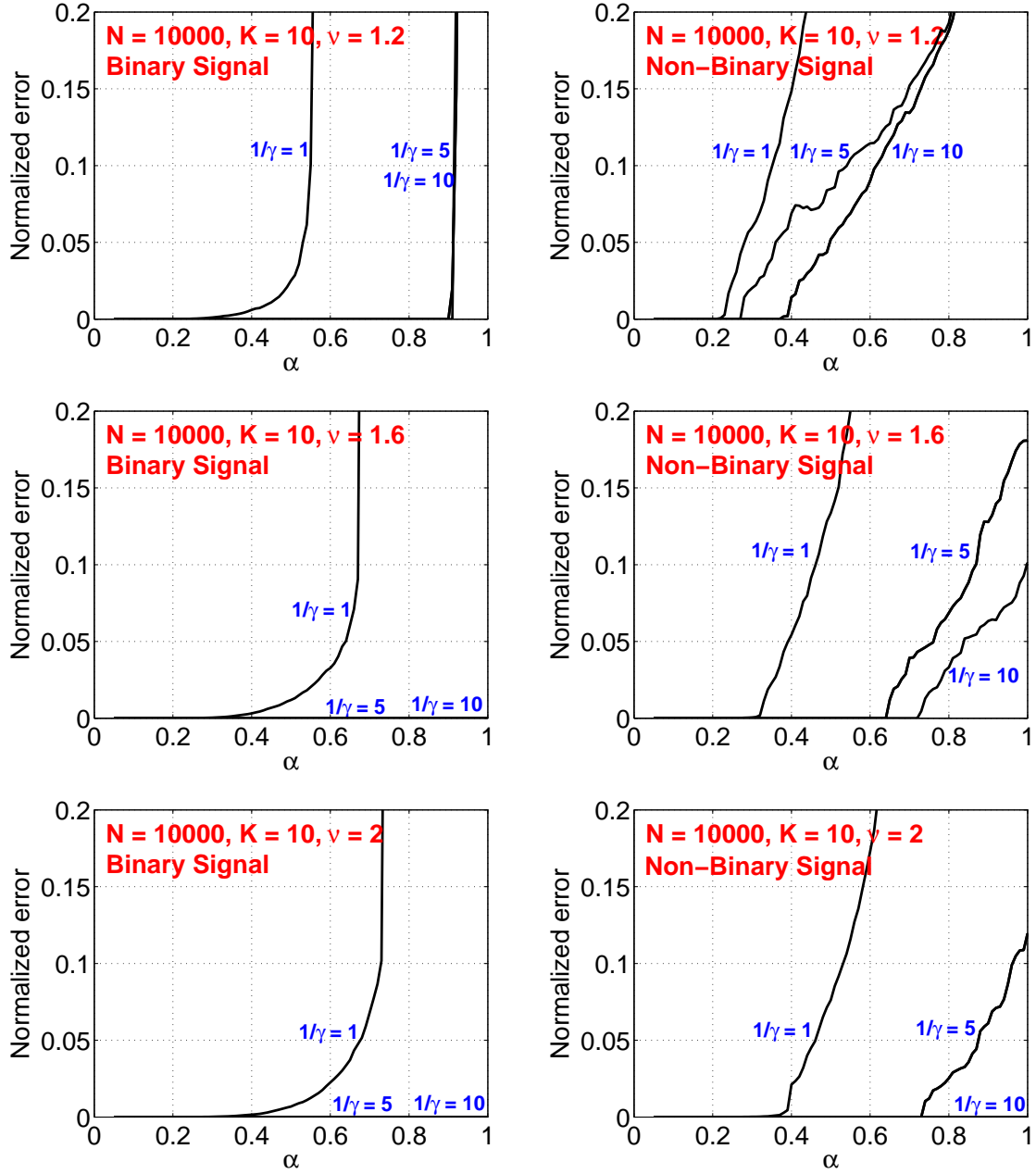


Figure 1: Normalized estimation errors (12) with $N = 10000$ and $K = 10$.

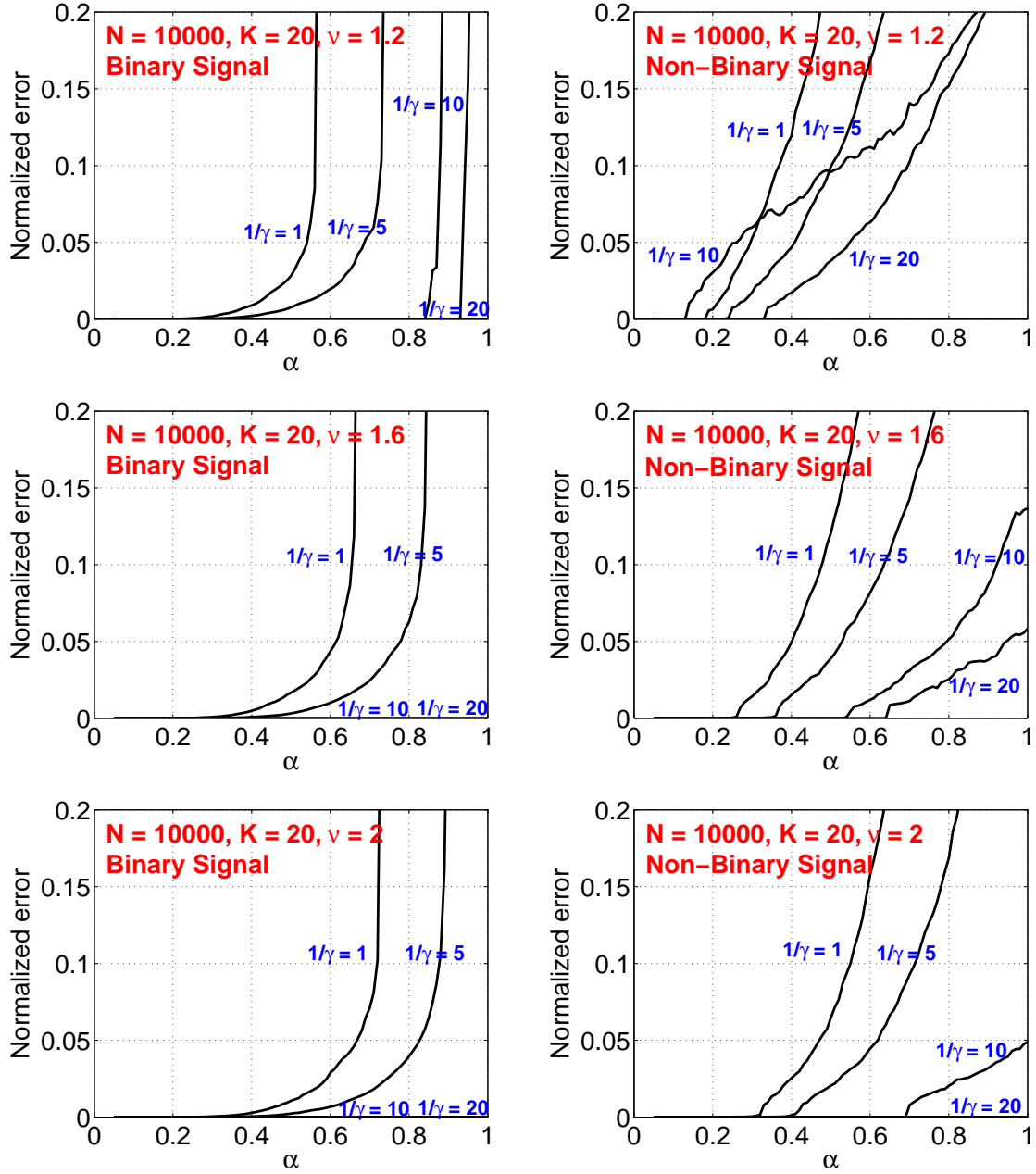


Figure 2: Normalized estimation errors (12) with $N = 10000$ and $K = 20$.

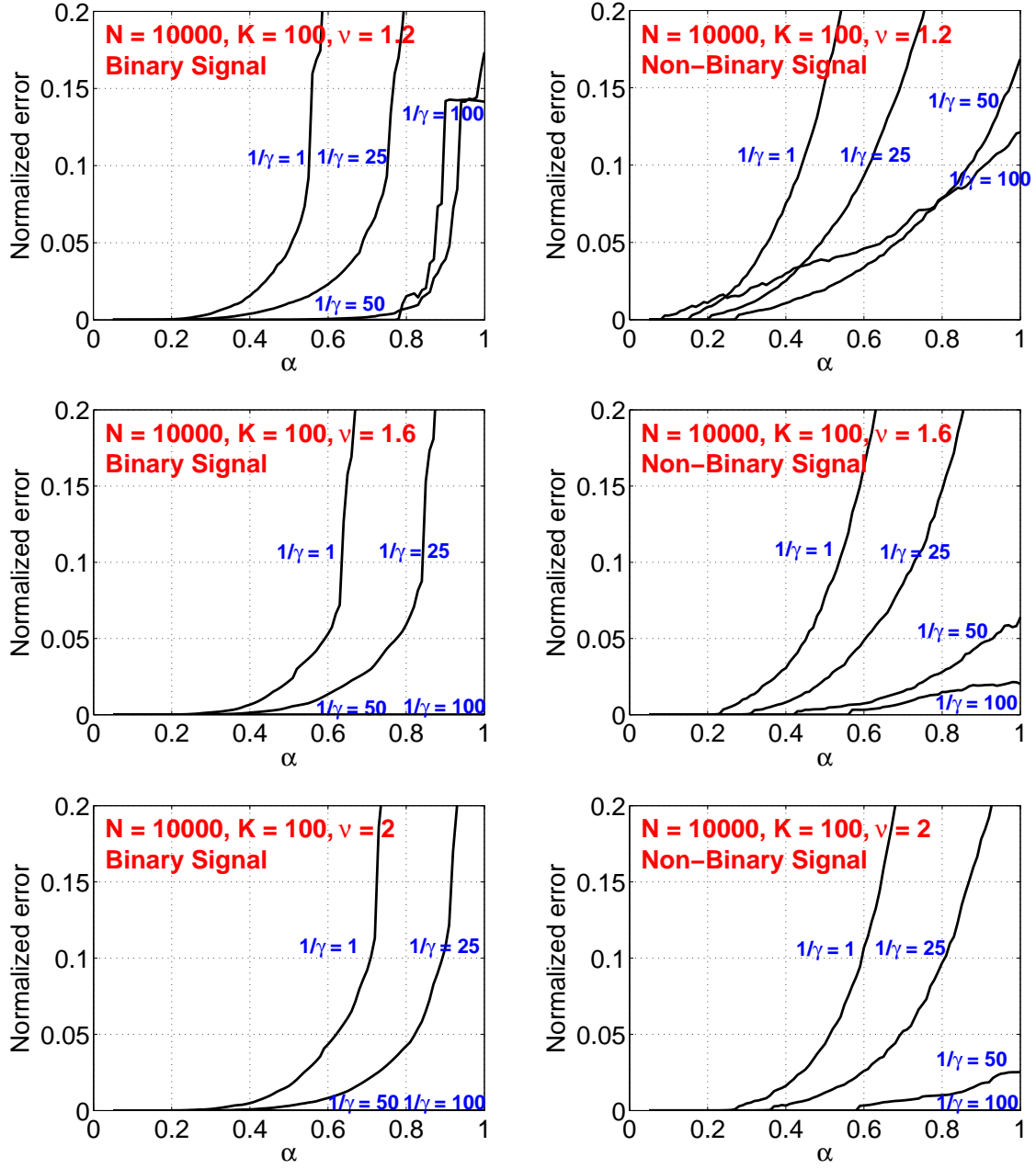


Figure 3: Normalized estimation errors (12) with $N = 10000$ and $K = 100$.

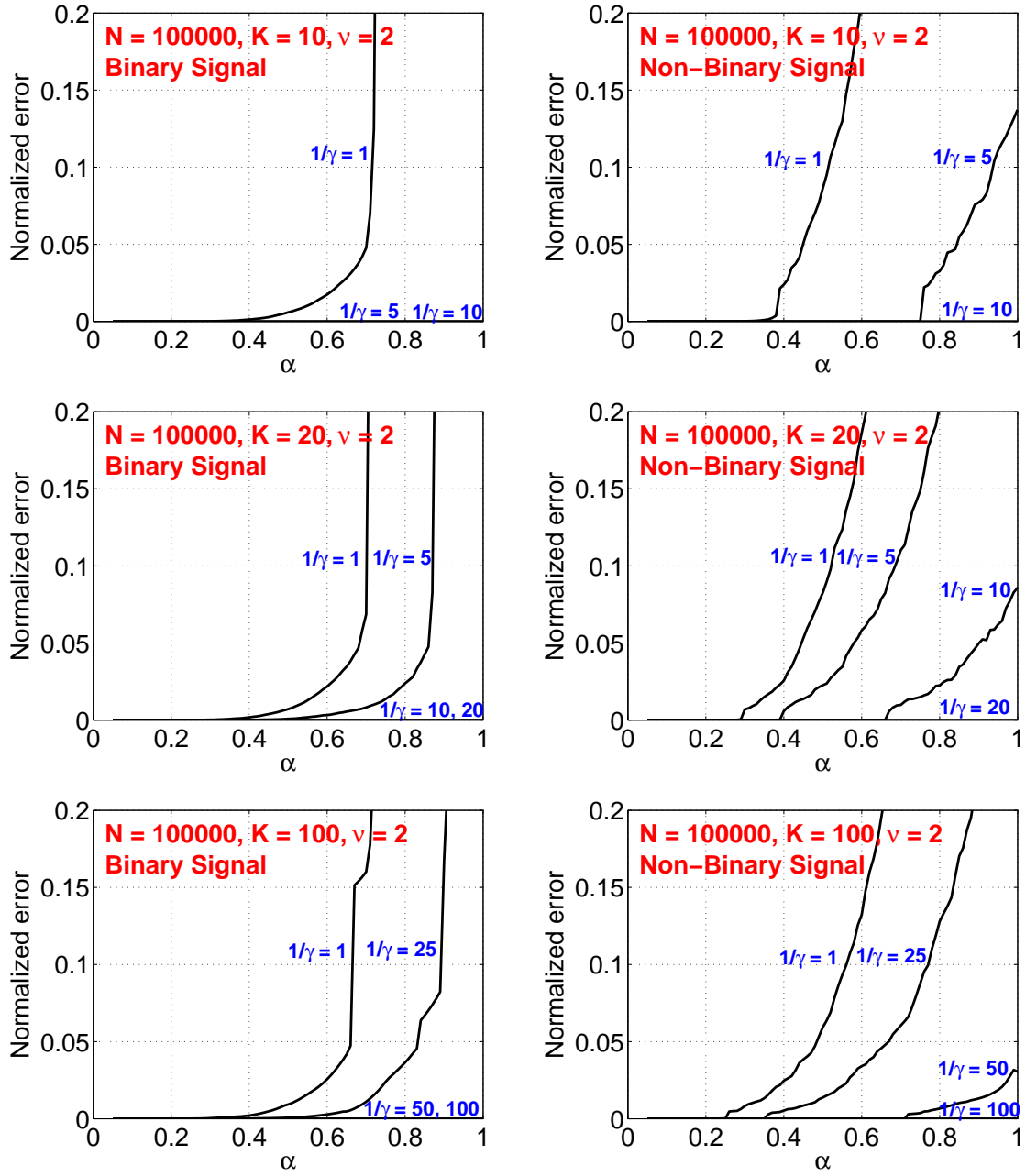


Figure 4: Normalized estimation errors (12) with $N = 100000$ and $\nu = 2$.

3 Analysis

Recall, we collect our measurements as

$$y_j = \sum_{i=1}^N x_i s_{ij} r_{ij}, \quad j = 1, 2, \dots, M \quad (13)$$

where $s_{ij} \sim S(\alpha, 1, 1)$ i.i.d. and

$$r_{ij} = \begin{cases} 1 & \text{with prob. } \gamma \\ 0 & \text{with prob. } 1 - \gamma \end{cases} \quad i.i.d. \quad (14)$$

And any s_{ij} and r_{ij} are also independent. Our proposed estimator is simply

$$\hat{x}_{i,min,\gamma} = \min_{j \in T_i} \frac{y_j}{s_{ij} r_{ij}} \quad (15)$$

where T_i is the set of nonzero entries in the i -th row of S , i.e.,

$$T_i = \{j, 1 \leq j \leq M, r_{ij} = 1\} \quad (16)$$

Conditional on $r_{ij} = 1$,

$$\frac{y_j}{s_{ij} r_{ij}} \Big|_{r_{ij}=1} = \frac{\sum_{t=1}^N x_t s_{tj} r_{tj}}{s_{ij}} = x_i + \frac{\sum_{t \neq i} x_t s_{tj} r_{tj}}{s_{ij}} = x_i + (\eta_{ij})^{1/\alpha} \frac{S_2}{S_1} \quad (17)$$

where $S_1, S_2 \sim S(\alpha, 1, 1)$, i.i.d., and

$$\eta_{ij} = \sum_{t \neq i}^N (x_t r_{tj})^\alpha = \sum_{t \neq i}^N x_t^\alpha r_{tj} \quad (18)$$

Note that

$$E(\eta_{ij}) = \gamma \sum_{t \neq i}^N x_t^\alpha \leq \gamma \sum_{t=1}^N x_t^\alpha, \quad \lim_{\alpha \rightarrow 0+} E(\eta_{ij}) \leq \gamma K \quad (19)$$

When the signals are binary, i.e., $x_i \in \{0, 1\}$, we have

$$\eta_{ij} \sim \begin{cases} \text{Binomial}(K, \gamma) & \text{if } x_i = 0 \\ \text{Binomial}(K-1, \gamma) & \text{if } x_i = 1 \end{cases} \quad (20)$$

The key in our theoretical analysis is the distribution of the ratio of two independent stable random variables. Here, we consider $S_1, S_2 \sim S(\alpha, 1, 1)$, i.i.d., and define

$$F_\alpha(t) = \mathbf{Pr} \left((S_2/S_1)^{\alpha/(1-\alpha)} \leq t \right), \quad t \geq 0 \quad (21)$$

There is a standard procedure to sample from $S(\alpha, 1, 1)$ [3]. We first generate an exponential random variable with mean 1, $w \sim \exp(1)$, and a uniform random variable $u \sim \text{unif}(0, \pi)$, and then compute

$$\frac{\sin(\alpha u)}{[\sin u \cos(\alpha\pi/2)]^{\frac{1}{\alpha}}} \left[\frac{\sin(u - \alpha u)}{w} \right]^{\frac{1-\alpha}{\alpha}} \sim S(\alpha, 1, 1) \quad (22)$$

Lemma 1 [11] For any $t \geq 0$, $S_1, S_2 \sim S(\alpha, 1, 1)$, i.i.d.,

$$F_\alpha(t) = \Pr \left((S_2/S_1)^{\alpha/(1-\alpha)} \leq t \right) = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \frac{1}{1 + Q_\alpha/t} du_1 du_2 \quad (23)$$

where

$$Q_\alpha = \left[\frac{\sin(\alpha u_2)}{\sin(\alpha u_1)} \right]^{\alpha/(1-\alpha)} \left[\frac{\sin u_1}{\sin u_2} \right]^{\frac{1}{1-\alpha}} \frac{\sin(u_2 - \alpha u_2)}{\sin(u_1 - \alpha u_1)} \quad (24)$$

In particular,

$$\lim_{\alpha \rightarrow 0+} F_\alpha(t) = \frac{1}{1 + 1/t}, \quad F_{0.5}(t) = \frac{2}{\pi} \tan^{-1} \sqrt{t} \quad \square \quad (25)$$

3.1 Error Probability

The following Lemma derives the general formula (26) for the error probability in terms of an expectation, which in general does not have a close-form solution. Nevertheless, when $\alpha = 0+$ and $\alpha = 0.5$, we can derive two convenient upper bounds, (28) and (30), respectively, which however are not tight.

Lemma 2

$$\Pr(\hat{x}_{i,min,\gamma} > x_i + \epsilon) = \left[1 - \gamma E \left\{ F_\alpha \left(\left(\frac{\epsilon^\alpha}{\eta_{ij}} \right)^{1/(1-\alpha)} \right) \right\} \right]^M \quad (26)$$

When $\alpha \rightarrow 0+$, we have

$$\Pr(\hat{x}_{i,min,\gamma} > x_i + \epsilon) \leq \left[1 - \frac{1}{1/\gamma + K - 1 + 1_{x_i=0}} \right]^M \quad (27)$$

$$\leq \left[1 - \frac{1}{1/\gamma + K} \right]^M \quad (28)$$

When $\alpha = 0.5$, we have

$$\Pr(\hat{x}_{i,min,\gamma} > x_i + \epsilon) \leq \left[1 - \gamma \frac{2}{\pi} \tan^{-1} \left(\frac{\sqrt{\epsilon}}{\gamma \sum_{t \neq i}^N x_t^{1/2}} \right) \right]^M \quad (29)$$

$$\leq \left[1 - \gamma \frac{2}{\pi} \tan^{-1} \left(\frac{\sqrt{\epsilon}}{\gamma \sum_{t=1}^N x_t^{1/2}} \right) \right]^M \quad (30)$$

Proof: See Appendix A. \square

It turns out, when $\alpha = 0+$, we can precisely evaluate the expectation (26) and derive an accurate complexity bound (31) in Lemma 3.

Lemma 3 As $\alpha \rightarrow 0+$, we have

$$\Pr(\hat{x}_{i,min,\gamma} > x_i + \epsilon) = \left[1 - \frac{1}{K + 1_{x_i=0}} (1 - (1 - \gamma)^{K+1_{x_i=0}}) \right]^M \quad (31)$$

$$\leq \left[1 - \frac{1}{K + 1} (1 - (1 - \gamma)^{K+1}) \right]^M \quad (32)$$

$$\leq \left[1 - \frac{1}{1/\gamma + K} \right]^M \quad (33)$$

Proof: See Appendix B. \square

3.2 Sample Complexity when $\alpha \rightarrow 0+$

Based on the precise error probability (31) in Lemma 3, we can derive the sample complexity bound from

$$(N - K) \left[1 - \frac{1}{K+1} (1 - (1 - \gamma)^{K+1}) \right]^M + K \left[1 - \frac{1}{K} (1 - (1 - \gamma)^K) \right]^M \leq \delta \quad (34)$$

Because $\left[1 - \frac{1}{K} (1 - (1 - \gamma)^K) \right]^M \leq \left[1 - \frac{1}{K+1} (1 - (1 - \gamma)^{K+1}) \right]^M$, it suffices to let

$$N \left[1 - \frac{1}{K+1} (1 - (1 - \gamma)^{K+1}) \right]^M \leq \delta$$

This immediately leads to the sample complexity result for $\alpha \rightarrow 0+$ in Theorem 1.

Theorem 1 *As $\alpha \rightarrow 0+$, the required number of measurements is*

$$M = \frac{1}{-\log \left[1 - \frac{1}{K+1} (1 - (1 - \gamma)^{K+1}) \right]} \log N/\delta \quad (35)$$

□

Remark: The required number of measurements (35) can essentially be written as

$$M = \frac{K}{1 - e^{-\gamma K}} \log N/\delta \quad (36)$$

The difference between (35) and (36) is very small even when K is small, as shown in Figure 5. Let $\lambda = \gamma K$. If $\lambda = 1$ (i.e., $\gamma = 1/K$), then the required M is about $1.58K \log N/\delta$. If $\lambda = 2$ (i.e., $\gamma = 2/K$), then M is about $1.16K \log N/\delta$. In other words, we can use a very sparse design matrix and the required number of measurements is only inflated slightly.

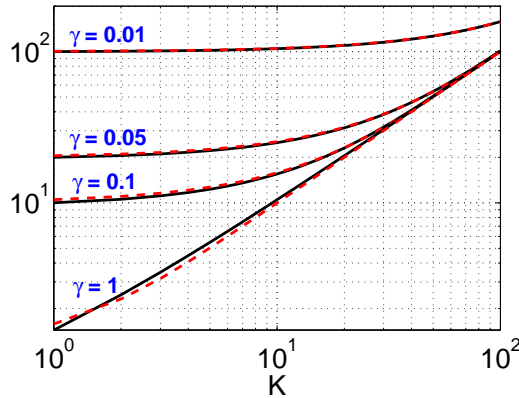


Figure 5: Solid curves: $\frac{1}{-\log \left[1 - \frac{1}{K+1} (1 - (1 - \gamma)^{K+1}) \right]}$. Dashed curves: $\frac{K}{1 - e^{-\gamma K}}$. The difference between (35) and (36) is very small even for small K . For large K , both terms approach K .

3.3 Worst-Case Sample Complexity

Theorem 2 *If we choose $\gamma = \frac{1}{K+1}$, then it suffices to choose the number of measurements by*

$$M = \frac{1}{-\log \left[1 - \frac{1}{K+1} \left(1 - \frac{1}{K+1} \right)^K \right]} \log N/\delta \quad (37)$$

Proof: See Appendix C. □

Remark: The worst-case complexity (37) can essentially be written as

$$M = eK \log N/\delta, \quad \text{if } \gamma = 1/K \quad (38)$$

where $e = 2.7183\dots$. The previous analysis of sample complexity for $\alpha \rightarrow 0+$ says that if $\gamma = 1/K$, it suffices to let $M = 1.58K \log N/\delta$, and if $\gamma = 2/K$, it suffices to let $M = 1.15K \log N/\delta$. This means that the worst-case analysis is quite conservative and the choice $\gamma = 1/K$ is not optimal for general $\alpha \in (0, 1)$.

Interestingly, it turns out that the worst-case sample complexity is attained when $\alpha \rightarrow 1-$.

3.4 Sample Complexity when $\alpha = 1-$

Theorem 3 *For a K -sparse signal whose nonzero coordinates are larger than ϵ , i.e., $x_i > \epsilon$ if $x_i > 0$. If we choose $\gamma = \frac{1}{K+1}$, as $\alpha \rightarrow 1-$, it suffices to choose the number of measurements by*

$$M = \frac{1}{-\log \left(1 - \frac{1}{K+1} \left(1 - \frac{1}{K+1} \right)^K \right)} \log N/\delta \quad (39)$$

Proof: The proof can be directly inferred from the proof of Theorem 2 at $\alpha = 1-$. □

Remark: Note that, if the assumption $x_i > \epsilon$ whenever $x_i > 0$ does not hold, then the required number of measurements will be smaller.

3.5 Sample Complexity Analysis for Binary Signals

As this point, we know the precise sample complexities for $\alpha = 0+$ and $\alpha = 1-$. And we also know the worst-case complexity. Nevertheless, it would be still interesting to study how the complexity varies as α changes between 0 and 1. While a precise analysis is difficult, we can perform an accurate analysis at least for binary signals, i.e., $x_i \in \{0, 1\}$. For convenience, we first re-write the general error probability as

$$\Pr(\hat{x}_{i,\min,\gamma} > x_i + \epsilon) = \left[1 - \frac{1}{K} (\gamma K) E \left\{ F_\alpha \left(\left(\frac{\epsilon^\alpha}{\eta_{ij}} \right)^{1/(1-\alpha)} \right) \right\} \right]^M \quad (40)$$

For binary signals, we have $\eta_{ij} \sim \text{Binomial}(K-1+1_{x_i=0}, \gamma)$. Thus, if $x_i = 0$, then

$$\begin{aligned} H &= H(\gamma, K; \epsilon, \alpha) \triangleq (\gamma K) E \left\{ F_\alpha \left(\left(\frac{\epsilon^\alpha}{\eta_{ij}} \right)^{1/(1-\alpha)} \right) \right\} \\ &= (\gamma K) \sum_{k=0}^K F_\alpha \left(\left(\frac{\epsilon^\alpha}{k} \right)^{1/(1-\alpha)} \right) \binom{K}{k} \gamma^k (1-\gamma)^{K-k} \end{aligned} \quad (41)$$

The required number of measurements can be written as $\frac{1}{-\log(1-H/K)} \log N/\delta$, or essentially $\frac{K}{H} \log N/\delta$. We can compute $H(\gamma, K; \epsilon, \alpha)$ for given γ, K, ϵ , and α , at least by simulations.

4 Poisson Approximation for Complexity Analysis with Binary Signals

Again, the purpose is to study more precisely how the sample complexity varies with $\alpha \in (0, 1)$, at least for binary signals. In this case, when $x_i = 0$, we have $\eta_{ij} \sim \text{Binomial}(K, \gamma)$. Elementary statistics tells us that we can well approximate this binomial with a Poisson distribution with parameter $\lambda = \gamma K$ especially when K is not small. Using the Poisson approximation, we can replace $H(\gamma, K; \epsilon, \alpha)$ in (41) by $h(\lambda; \epsilon, \alpha)$ and re-write the error probability as

$$\Pr(\hat{x}_{i, \min, \gamma} > x_i + \epsilon) = \left[1 - \frac{1}{K} h(\lambda; \epsilon, \alpha) \right]^M \quad (42)$$

where

$$\begin{aligned} h(\lambda; \epsilon, \alpha) &= \lambda \sum_{k=0}^{\infty} F_{\alpha} \left(\left(\frac{\epsilon^{\alpha}}{k} \right)^{1/(1-\alpha)} \right) \frac{e^{-\lambda} \lambda^k}{k!} \\ &= \lambda e^{-\lambda} + \lambda e^{-\lambda} \sum_{k=1}^{\infty} F_{\alpha} \left(\left(\frac{\epsilon^{\alpha}}{k} \right)^{1/(1-\alpha)} \right) \frac{\lambda^k}{k!} \end{aligned} \quad (43)$$

which can be computed numerically for any given λ and ϵ .

The required number of measurements can be computed from

$$N \left[1 - \frac{1}{K} h(\lambda; \epsilon, \alpha) \right]^M = \delta \iff M = \frac{\log N/\delta}{-\log \left[1 - \frac{1}{K} h(\lambda; \epsilon, \alpha) \right]} \quad (44)$$

for which it suffices to choose M such that

$$M = \frac{K}{h(\lambda; \epsilon, \alpha)} \log N/\delta \quad (45)$$

Therefore, we hope $h(\lambda; \epsilon, \alpha)$ should be as large as possible.

4.1 Analysis for $\alpha = 0.5$

Before we demonstrate the results via Poisson approximation for general $0 < \alpha < 1$, we would like to illustrate the analysis particularly for $\alpha = 0.5$, which is a case readers can more easily verify.

Recall when $\alpha = 0.5$, the error probability can be written as

$$\Pr(\hat{x}_{i, \min, \gamma} > x_i + \epsilon) = \left[1 - \frac{1}{K} (\gamma K) E \left\{ \frac{2}{\pi} \tan^{-1} \left(\left(\frac{\sqrt{\epsilon}}{\eta_{ij}} \right) \right) \right\} \right]^M = \left[1 - \frac{1}{K} H(\gamma, K; \epsilon, 0.5) \right]^M$$

where

$$H(\gamma, K; \epsilon, 0.5) = (\gamma K) \frac{2}{\pi} \sum_{k=0}^K \tan^{-1} \left(\frac{\sqrt{\epsilon}}{k} \right) \binom{K}{k} \gamma^k (1-\gamma)^{K-k} \quad (46)$$

From Lemma 2, in particular (30), we know there is a convenient lower bound of H :

$$H(\gamma, K; \epsilon, 0.5) \geq H^{\text{lower}}(\gamma, K; \epsilon, 0.5) = (\gamma K) \left\{ \frac{2}{\pi} \tan^{-1} \left(\frac{\sqrt{\epsilon}}{\gamma K} \right) \right\} = \lambda \frac{2}{\pi} \tan^{-1} \left(\frac{\sqrt{\epsilon}}{\lambda} \right) \quad (47)$$

We will compare the precise $H(\gamma, K; \epsilon, 0.5)$ with its lower bound $H^{lower}(\gamma, K; \epsilon, 0.5)$, along with the Poisson approximation:

$$H(\gamma, K; \epsilon, 0.5) \approx h(\lambda; \epsilon, 0.5) = \lambda e^{-\lambda} \frac{2}{\pi} \sum_{k=0}^{\infty} \tan^{-1} \left(\frac{\sqrt{\epsilon}}{k} \right) \frac{\lambda^k}{k!} \quad (48)$$

Figure 6 confirms that the Poisson approximation is very accurate unless K is very small, while the lower bound is conservative especially when γ is around the optimal value. For small ϵ , the optimal γ is around $1/K$, which is consistent with the general worst-case complexity result.

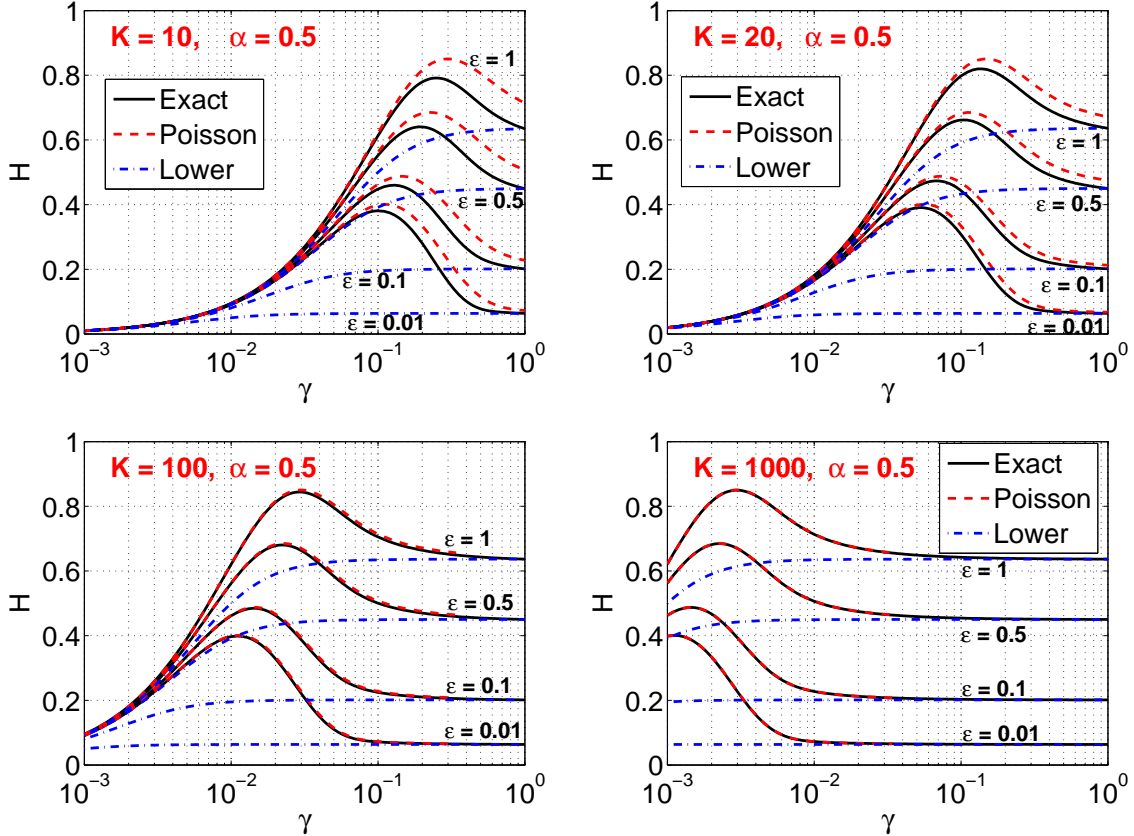


Figure 6: $H(\gamma, K; \epsilon, 0.5)$ at four different values of $\epsilon \in \{0.01, 0.1, 0.5, 1\}$. The exact H and its Poisson approximation $h(\lambda; \epsilon, 0.5)$ match very well unless K is very small. The lower bound of H is conservative, especially when γ is around the optimal value. For small ϵ , the optimal γ is around $1/K$.

4.2 Poisson Approximation for General $0 < \alpha < 1$

Once we are convinced that the Poisson approximation is reliable at least for $\alpha = 0.5$, we can use this tool to study for general $\alpha \in (0, 1)$. Again, assume the Poisson approximation, we have

$$\Pr(\hat{x}_{i,min,\gamma} > x_i + \epsilon) = \left[1 - \frac{1}{K} h(\lambda; \epsilon, \alpha) \right]^M$$

where

$$h(\lambda; \epsilon, \alpha) = \lambda e^{-\lambda} + \lambda e^{-\lambda} \sum_{k=1}^{\infty} F_{\alpha} \left(\left(\frac{\epsilon^{\alpha}}{k} \right)^{1/(1-\alpha)} \right) \frac{\lambda^k}{k!}$$

The required number of measurements can be computed from $M = \frac{K}{h(\lambda; \epsilon, \alpha)} \log N/\delta$.

As shown in Figure 7, at fixed ϵ and λ , the optimal (highest) h is larger when α is smaller. The optimal h occurs at larger λ when α is closer to zero and at smaller λ when α is closer to 1.

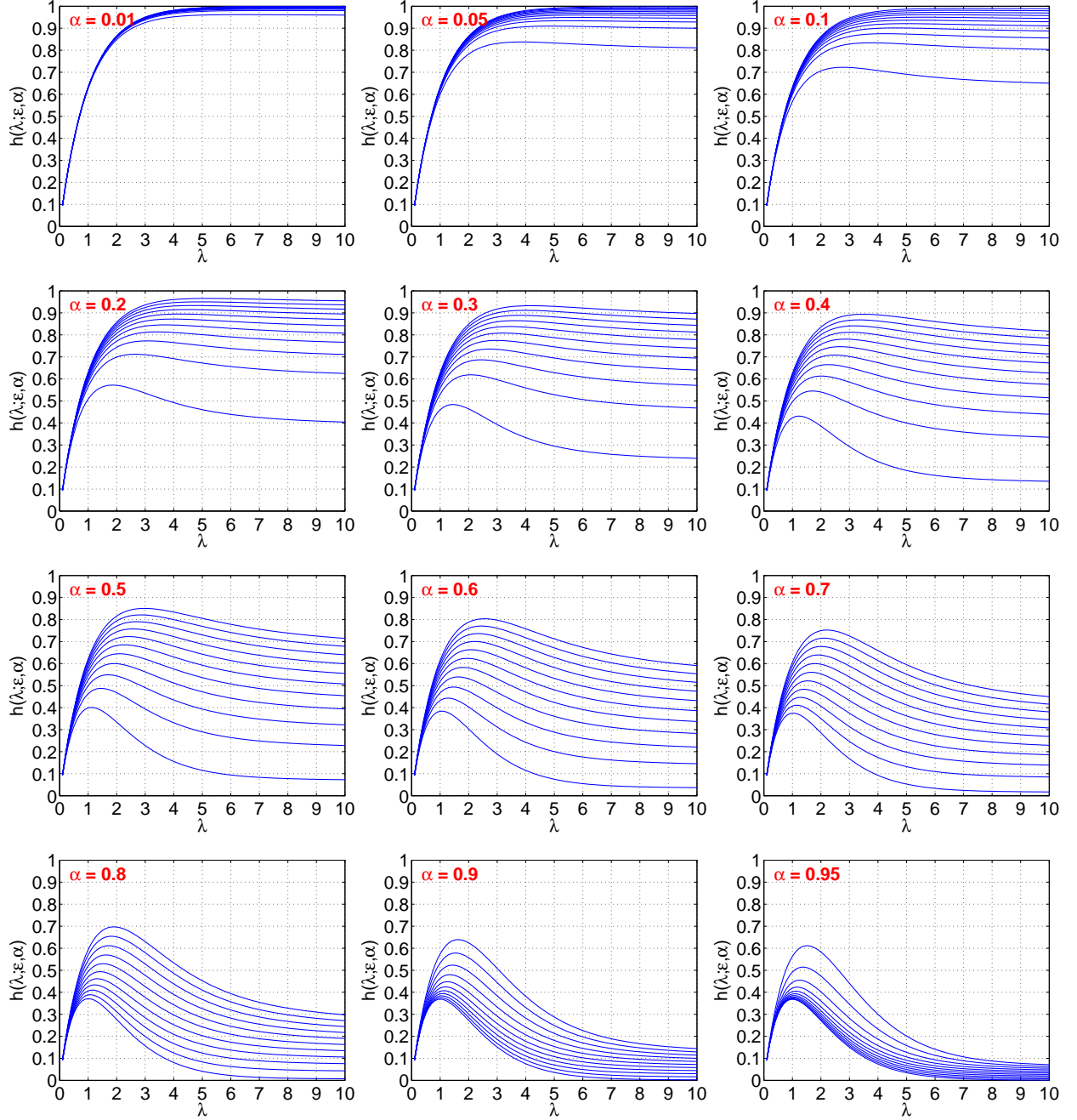


Figure 7: $h(\lambda; \epsilon, \alpha)$ as defined in (43) for selected α values ranging from 0.01 to 0.95. In each panel, each curve corresponds to an ϵ value, where $\epsilon \in \{0.01, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$ (from bottom to top). In each panel, the curve for $\epsilon = 0.01$ is the lowest and the curve for $\epsilon = 1$ is the highest.

Figure 8 plots the optimal (smallest) $1/h(\lambda; \epsilon, \alpha)$ values (left panel) and the optimal λ values (right panel) which achieve the optimal h .

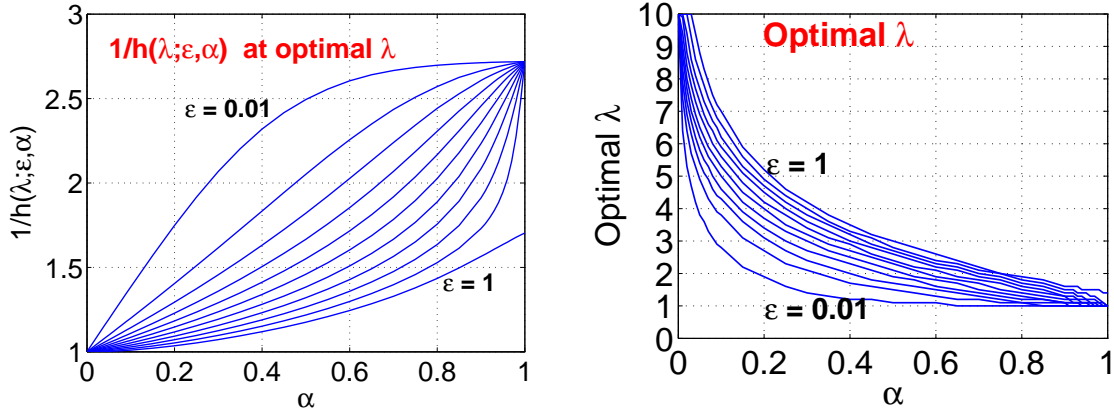


Figure 8: Left Panel: $1/h(\lambda; \epsilon, \alpha)$ at the optimal λ values. Right Panel: the optimal λ values.

Figure 9 plots $1/h(\lambda; \epsilon, \alpha)$ for fixed $\lambda = 1$ (left panel) and $\lambda = 2$ (right panel), together with the optimal $1/h(\lambda; \epsilon, \alpha)$ values (dashed curves).

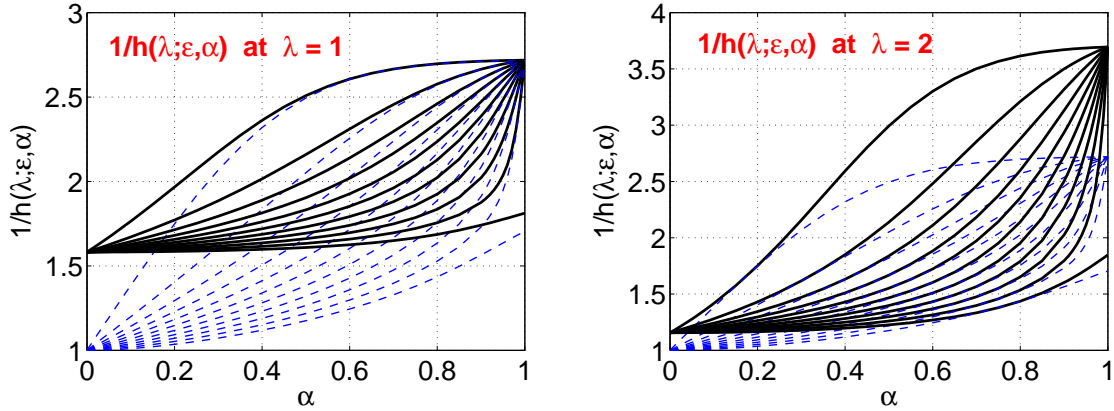


Figure 9: $1/h(\lambda; \epsilon, \alpha)$ at the fixed $\lambda = 1$ (left panel) and $\lambda = 2$ (right panel). The dashed curves correspond to $1/h(\lambda; \epsilon, \alpha)$ at the optimal λ values.

4.3 Poisson Approximation for $\alpha \rightarrow 1-$

We now examine $h(\lambda; \epsilon, \alpha)$ closely at $\alpha = 1-$, i.e., $\frac{1}{1-\alpha} \rightarrow \infty$.

$$h(\lambda; \epsilon, \alpha) = \lambda e^{-\lambda} + \lambda e^{-\lambda} \sum_{k=1}^{\infty} F_{\alpha} \left(\left(\frac{\epsilon^{\alpha}}{k} \right)^{1/(1-\alpha)} \right) \frac{\lambda^k}{k!}$$

Interestingly, when $\epsilon = 1$, only $k = 0$ and $k = 1$ will be useful, because otherwise $\left(\frac{\epsilon^{\alpha}}{k} \right)^{1/(1-\alpha)} \rightarrow \infty$ as $\Delta = 1 - \alpha \rightarrow 0$. When $\epsilon < 1$, then only $k = 0$ is useful. Thus, we can write

$$h(\lambda; \epsilon < 1, \alpha = 1-) = \lambda e^{-\lambda} \tag{49}$$

$$h(\lambda; \epsilon = 1, \alpha = 1-) = \lambda e^{-\lambda} + \lambda^2 e^{-\lambda} F_{1-}(1) = \lambda e^{-\lambda} + \lambda^2 e^{-\lambda} / 2 \tag{50}$$

Notes that $F_{1-}(1) = 1/2$ due to symmetry.

This mean, the maximum of $h(\lambda; \epsilon < 1, \alpha = 1-)$ is e^{-1} attained at $\lambda = 1$, and the maximum of $h(\lambda; \epsilon = 1, \alpha = 1-)$ is $e^{-\sqrt{2}}(1 + \sqrt{2}) = 0.5869$, attained at $\lambda = \sqrt{2}$, as confirmed by Figure 10. In other words, it suffices to choose the number of measurements to be

$$M = eK \log N/\delta \quad \text{if } \epsilon < 1, \quad M = 1.7038K \log N/\delta \quad \text{if } \epsilon = 1 \quad (51)$$

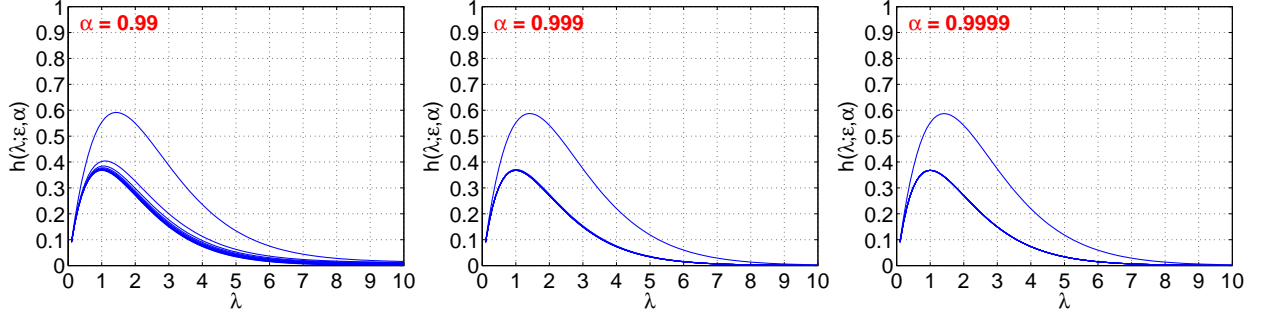


Figure 10: $h(\lambda; \epsilon, \alpha)$ as defined in (43) for α close to 1. As $\alpha \rightarrow 1-$, the maximum of $h(\lambda; \epsilon, \alpha)$ approaches e^{-1} attained at $\lambda = 1$, for all $\epsilon < 1$. When $\epsilon = 1$, the maximum approaches 0.5869 , attained at $\lambda = \sqrt{2}$.

5 Conclusion

In this paper, we extend the prior work on *Compressed Counting meets Compressed Sensing* [11] and *very sparse stable random projections* [9, 6] to the interesting problem of sparse recovery of nonnegative signals. The design matrix is highly sparse in that on average only γ -fraction of the entries are nonzero; and we sample the nonzero entries from an α -stable maximally-skewed distribution where $\alpha \in (0, 1)$. Our theoretical analysis demonstrates that the design matrix can be extremely sparse, e.g., $\gamma = \frac{1}{K} \sim \frac{2}{K}$. In fact, when α is away from 0, it is much more preferable to use a very sparse design.

A Proof of Lemma 2

$$\begin{aligned}
& \Pr(\hat{x}_{i,min,\gamma} > x_i + \epsilon) \\
&= E \left(\Pr \left(\frac{y_j}{s_{ij}} > x_i + \epsilon, j \in T_i | T_i \right) \right) \\
&= E \prod_{j \in T_i} \left[\Pr \left(\frac{S_2}{S_1} > \frac{\epsilon}{\eta_{ij}^{1/\alpha}} \right) \right] \\
&= E \prod_{j \in T_i} \left[1 - F_\alpha \left(\left(\frac{\epsilon^\alpha}{\eta_{ij}} \right)^{1/(1-\alpha)} \right) \right] \\
&= E \left\{ \left[1 - E \left\{ F_\alpha \left(\left(\frac{\epsilon^\alpha}{\eta_{ij}} \right)^{1/(1-\alpha)} \right) \right\} \right]^{|T_i|} \right\} \\
&= \left[1 - \gamma + \gamma \left\{ 1 - E \left\{ F_\alpha \left(\left(\frac{\epsilon}{\eta_{ij}} \right)^{\alpha/(1-\alpha)} \right) \right\} \right\} \right]^M \\
&= \left[1 - \gamma E \left\{ F_\alpha \left(\left(\frac{\epsilon^\alpha}{\eta_{ij}} \right)^{1/(1-\alpha)} \right) \right\} \right]^M
\end{aligned}$$

When $\alpha = 0.5$, we have $F_\alpha(t) = \frac{2}{\pi} \tan^{-1} \sqrt{t}$ and hence

$$\begin{aligned}
& \Pr(\hat{x}_{i,min,\gamma} > x_i + \epsilon) \\
&= \left[1 - \gamma E \left\{ F_\alpha \left(\left(\frac{\epsilon^\alpha}{\eta_{ij}} \right)^{1/(1-\alpha)} \right) \right\} \right]^M \\
&= \left[1 - \gamma E \left\{ \frac{2}{\pi} \tan^{-1} \left(\left(\frac{\sqrt{\epsilon}}{\eta_{ij}} \right) \right) \right\} \right]^M \\
&\leq \left[1 - \gamma \left\{ \frac{2}{\pi} \tan^{-1} \left(\left(\frac{\sqrt{\epsilon}}{E\eta_{ij}} \right) \right) \right\} \right]^M \quad (\text{Jensen's Inequality}) \\
&\leq \left[1 - \gamma \left\{ \frac{2}{\pi} \tan^{-1} \left\{ \frac{1}{\gamma} \frac{\sqrt{\epsilon}}{\sum_{t \neq i} x_t^{1/2}} \right\} \right\} \right]^M
\end{aligned}$$

When $\alpha = 0+$, we have $F_{0+}(t) = \frac{1}{1+1/t}$ and hence

$$\begin{aligned}
& \Pr(\hat{x}_{i,min,\gamma} > x_i + \epsilon) \\
&= \lim_{\alpha \rightarrow 0+} \left[1 - \gamma E \left\{ F_{0+} \left(\frac{1}{\eta_{ij}} \right) \right\} \right]^M \\
&= \lim_{\alpha \rightarrow 0+} \left[1 - \gamma E \left\{ \left(\frac{1}{1 + \eta_{ij}} \right) \right\} \right]^M \\
&\leq \lim_{\alpha \rightarrow 0+} \left[1 - \gamma \left\{ \left(\frac{1}{1 + E\eta_{ij}} \right) \right\} \right]^M \\
&\leq \lim_{\alpha \rightarrow 0+} \left[1 - \gamma \frac{1}{1 + \gamma K} \right]^M \\
&= \left[1 - \frac{1}{1/\gamma + K} \right]^M
\end{aligned}$$

This completes the proof.

B Proof of Lemma 3

Proof: When $\alpha = 0+$, we have $F_{0+}(t) = \frac{1}{1+1/t}$ and hence

$$\Pr(\hat{x}_{i,min,\gamma} > x_i + \epsilon) = \lim_{\alpha \rightarrow 0+} \left[1 - \gamma E \left\{ \left(\frac{1}{1 + \eta_{ij}} \right) \right\} \right]^M$$

Suppose $x_i = 0$, then as $\alpha \rightarrow 0+$, $\eta_{ij} \sim \text{Binomial}(K, \gamma)$, and

$$\begin{aligned}
& E \left(\frac{1}{1 + \eta_{ij}} \right) \\
&= \sum_{n=0}^K \frac{1}{1+n} \binom{K}{n} \gamma^n (1-\gamma)^{K-n} \\
&= \sum_{n=0}^K \frac{1}{1+n} \frac{K!}{n!(K-n)!} \gamma^n (1-\gamma)^{K-n} \\
&= \sum_{n=0}^K \frac{K!}{(n+1)!(K-n)!} \gamma^n (1-\gamma)^{K-n} \\
&= \frac{1}{K+1} \frac{1}{\gamma} \sum_{n=0}^K \frac{(K+1)!}{(n+1)!((K+1)-(n+1))!} \gamma^{n+1} (1-\gamma)^{(K+1)-(n+1)} \\
&= \frac{1}{K+1} \frac{1}{\gamma} \sum_{n=1}^{K+1} \frac{(K+1)!}{(n)!((K+1)-(n))!} \gamma^n (1-\gamma)^{(K+1)-(n)} \\
&= \frac{1}{K+1} \frac{1}{\gamma} \left\{ \sum_{n=0}^{K+1} \frac{(K+1)!}{(n)!((K+1)-(n))!} \gamma^n (1-\gamma)^{(K+1)-(n)} - (1-\gamma)^{K+1} \right\} \\
&= \frac{1}{K+1} \frac{1}{\gamma} \{ 1 - (1-\gamma)^{K+1} \}
\end{aligned}$$

Similarly, suppose $x_i > 0$, we have

$$E\left(\frac{1}{1+\eta_{ij}}\right) = \frac{1}{K} \frac{1}{\gamma} \{1 - (1-\gamma)^K\}$$

Therefore, as $\alpha \rightarrow 0+$, when $x_i = 0$, we have

$$\Pr(\hat{x}_{i,min,\gamma} > x_i + \epsilon) = \left[1 - \frac{1}{K+1} (1 - (1-\gamma)^{K+1})\right]^M$$

and when $x_i > 0$, we have

$$\Pr(\hat{x}_{i,min,\gamma} > x_i + \epsilon) = \left[1 - \frac{1}{K} (1 - (1-\gamma)^K)\right]^M$$

To conclude the proof, we need to show

$$\begin{aligned} & \left[1 - \frac{1}{K+1} (1 - (1-\gamma)^{K+1})\right]^M \leq \left[1 - \frac{1}{1/\gamma + K}\right]^M \\ \iff & \frac{1}{K+1} (1 - (1-\gamma)^{K+1}) \geq \frac{1}{1/\gamma + K} \\ \iff & h(\gamma, K) = 1/\gamma - (1-\gamma)^{K+1}/\gamma - K(1-\gamma)^{K+1} - 1 \geq 0 \end{aligned}$$

Note that $0 \leq \gamma \leq 1$, $h(0, K) = h(1, K) = h(\gamma, 1) = 0$. Furthermore

$$\begin{aligned} \frac{\partial h(\gamma, K)}{\partial K} &= - (1-\gamma)^{K+1} \log(1-\gamma)/\gamma - (1-\gamma)^{K+1} - K(1-\gamma)^{K+1} \log(1-\gamma) \\ &= - (1-\gamma)^{K+1} (\log(1-\gamma)/\gamma + 1 + K \log(1-\gamma)) \geq 0 \end{aligned}$$

as $\log(1-\gamma)/\gamma < -1$. Thus, $h(\gamma, K)$ is a monotonically increasing function of K and this completes the proof.

C Proof of Theorem 2

$$\begin{aligned} \Pr(\hat{x}_{i,min,\gamma} > x_i + \epsilon) &= \left[1 - \gamma E\left\{F_\alpha\left(\left(\frac{\epsilon^\alpha}{\eta_{ij}}\right)^{1/(1-\alpha)}\right)\right\}\right]^M \\ &\geq [1 - \gamma \Pr(\eta_{ij} = 0)]^M \\ &= [1 - \gamma (1-\gamma)^{K-1+1_{x_i=0}}]^M \\ &\geq [1 - \gamma (1-\gamma)^K]^M \end{aligned}$$

The minimum of $\gamma (1-\gamma)^{K-1+1_{x_i=0}}$ is attained at $\gamma = \frac{1}{K+1+1_{x_i=0}}$. If we choose $\gamma^* = \frac{1}{K+1}$, then

$$\Pr(\hat{x}_{i,min,\gamma} > x_i + \epsilon) \geq [1 - \gamma^* (1-\gamma^*)^K]^M = \left[1 - \frac{1}{K+1} \left(1 - \frac{1}{K+1}\right)^K\right]^M$$

and it suffices to choose M so that

$$M = \frac{1}{-\log\left[1 - \frac{1}{K+1} \left(1 - \frac{1}{K+1}\right)^K\right]} \log N/\delta$$

This completes the proof.

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